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LETTER TO THE EDITOR

Mean number of clusters for percolation processes in two dimensions

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Abstract. We use a transformation suggested recently by Zwanzig and Ramshaw on the series expansion for the mean number of clusters in the square bond and triangular site percolation problems. The new series are smooth and well behaved, and we estimate critical exponents and amplitudes. A value of -0.668 ± 0.004 is found for the exponent α_P, which is close to the rational fraction $-2/3$.

In a recent paper Zwanzig and Ramshaw (1976) reported an analysis of free energy series for the q-state Potts model on the square (SQ) lattice, in which they made use of a change of variable based on the known duality relation for this model (Potts 1952). If x is the low-temperature expansion variable ($x=\exp(-\epsilon/kT)$), this relation may be expressed in the form $B(x)=B(t)$, where B is a function closely related to the free energy, and t is the high-temperature expansion variable, related to x by

$$x=\frac{1-t}{1+(q-1)t};\qquad t=\frac{1-x}{1+(q-1)x}.$$

For the particular case $q=2$ (the spin-$\frac{1}{2}$ Ising model), B is known to be a function of xt ($=u$, say). This suggested as a natural extension the use of u as a variable for $q>2$. The resulting series expansions for the free energy were found to possess coefficients of uniform sign, and the use of the ratio method yielded values for the exponent α which were monotonically increasing with q.

The Potts and spin-$\frac{1}{2}$ Ising models have been shown by Kasteleyn and Fortuin (1969) to be special cases of a 'random cluster model', from which they are obtained by varying an integral parameter q: $q>2$ corresponds to the Potts and $q=2$ to the Ising model. These latter authors point out that setting q equal to unity results in the bond percolation model, with the mean number of clusters corresponding to the free energy of the other models. (For definitions see for example Essam 1972.) The relevant expansion variables are in this case the bond occupation probability p (corresponding to the high-temperature variable t) and $Q=1-p$ (corresponding to the low-temperature variable x). (We use capital Q to prevent confusion with q above.) For the SQ lattice the model is self-dual, and the critical probability is known to be exactly 0·5 (Sykes and Essam 1964). The low density series expansion for the mean number of clusters $k_L(p)$ is found to alternate in sign from the term in p^6 onwards, suggesting the existence of a dominating singularity on the negative real axis. This is supported by Padé analysis, which indicates the presence of a singularity at $p=-0.41\pm0.02$. For this reason it has been difficult to obtain reliable estimates of critical behaviour.

Sykes and Essam (1964) have shown that $k_L(p)$ is related to the high density series expansion $k_H(Q)$ by $k_L(p) = \phi(p) + k_H(p)$ where, for the SQ bond problem, $\phi(p)$ is a cubic polynomial. If we express this duality relation in terms of the function $B(p) = k_L(p) - \frac{1}{2}\phi(p)$, equal to half the total number of clusters irrespective of colour, it becomes $B(p) = B(1-p) = B(Q)$. Thus the suggested new expansion variable is $u = p(1-p) = pQ$, and we put $B(p) = \bar{B}(u) = \Sigma_{n=1} a_n u^n$.

The transformation thus defined maps the critical probability p_c to $u_c = 0{\cdot}25$, and the point $p = -0{\cdot}41$ to $u \simeq -0{\cdot}58$; hence the series for $\bar{B}(u)$ should be dominated ultimately by the singularity of interest. If the singular part of $k_L(p)$ has the dominant form $|p_c - p|^{2-\alpha_P}$, $B(p)$ will be an even function of $(p_c - p)$, and $\bar{B}(u)$ an even function of $(u_c - u)$, with singular part $|u_c - u|^{1-\alpha_P/2}$. Because of the self-duality of the model the low and high density exponents α_P and α'_P are equal.

Using the series expansion for $k_L(p)$ given by Sykes and Essam (1964) and extended to 19 terms by Sykes (private communication), we have obtained the series for $\bar{B}(u)$ for the SQ bond problem. The coefficients in this series are of uniform sign and are sufficiently smooth for the ratio method to be useful in extrapolation. To estimate the exponent $1-\alpha_P/2$ of the singularity at u_c, we have calculated the first-order Neville table extrapolants g_n to the quantity

$$h_n = 1 + n(u_c a_n / a_{n-1} - 1) \sim -1 + \alpha_P/2 + O(1/n).$$

These are expected to behave like $g_n = -1 + \alpha_P/2 + O(1/n^2)$. In figure 1 we show this quantity plotted against $1/n^2$: in such a graph it should tend linearly to $-1+\alpha_P/2$ as n goes to infinity. The graph in fact shows damped oscillations, characteristic of the negative real axis singularity; nevertheless we can estimate the extrapolant to be in the range $-1{\cdot}34 \pm 0{\cdot}01$, which corresponds to $\alpha_P = -0{\cdot}68 \pm 0{\cdot}02$. This represents a significant improvement in precision over the earlier estimate of $-0{\cdot}7 \pm 0{\cdot}2$ based on Padé analysis of the untransformed series, due to Harris *et al* (1975).

One other two-dimensional percolation problem is also known to be self-dual: namely the site problem on the triangular lattice (Sykes and Essam 1964). We have applied the same transformation to the 22 term series obtained by Sykes (private communication) for this model, and have plotted the corresponding quantity g_n against $1/n^2$ in figure 2. The behaviour is similar to that for the SQ bond problem, but the convergence appears to be more rapid (as expected for a close packed lattice) and we estimate $-1+\alpha_P/2$ to be in the range $-1{\cdot}334 \pm 0{\cdot}002$, i.e. $\alpha_P = -0{\cdot}668 \pm 0{\cdot}004$.

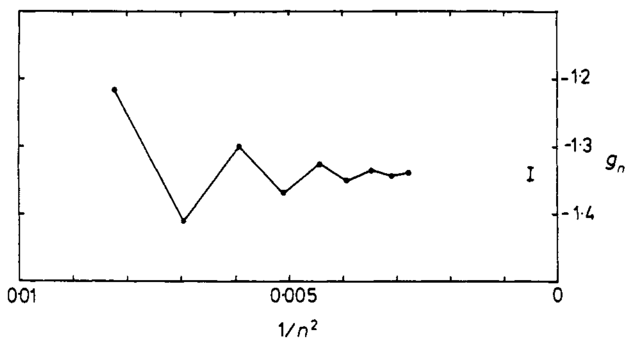

Figure 1. First-order Neville table extrapolants g_n for $-1+\alpha_P/2$, plotted against $1/n^2$ for the SQ bond problem.

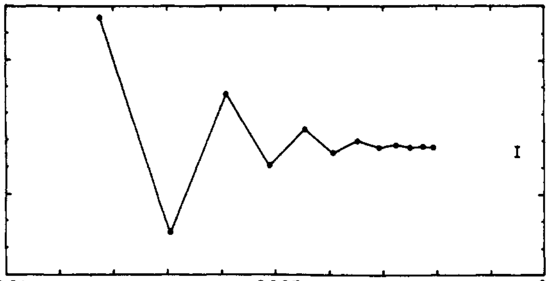

Figure 2. First-order Neville table extrapolants g_n for $-1+\alpha_P/2$, plotted against $1/n^2$ for the triangular lattice site problem.

Our results are consistent with α_P having the same value for both models; if this value is a simple rational fraction, then $\alpha_P = \alpha'_P = -2/3$ is obviously suggested. This estimate for α'_P, together with that for the percolation probability exponent β_P, obtained by Sykes *et al* (1976), can be used to place a lower limit on the high density mean size exponent γ'_P. Series expansions for the latter have proved extremely difficult to extrapolate (Sykes *et al* 1976). However, substitution of $\alpha'_P = -0{\cdot}668 \pm 0{\cdot}004$ and $\beta_P = 0{\cdot}138 \pm 0{\cdot}007$ into the Rushbrooke–Kasteleyn–Fortuin inequality $\alpha'_P + 2\beta_P + \gamma'_P \geq 2$ yields $\gamma'_P \geq 2{\cdot}38 \pm 0{\cdot}02$.

To estimate critical amplitudes, we assume that α_P is exactly $-2/3$ and employ direct extrapolation and summation techniques to the transformed series. We find that this is well represented by the form

$$\bar{B}(u) = n_c + C|u_c - u| + D|u_c - u|^{4/3} + \ldots$$

i.e.

$$k_L(p) = n_c + A(p_c - p) + C(p_c - p)^2 + D|p_c - p|^{8/3} + \ldots.$$

The value of A is obtainable exactly from the matching polynomial $\phi(p)$, and this and the other amplitudes are found to be as follows in table 1.

Table 1.

	SQ bond	Triangular site
n_c	$0{\cdot}0173 \pm 0{\cdot}0003$	$0{\cdot}0168 \pm 0{\cdot}0002$
A	$0{\cdot}25$	$0{\cdot}25$
C	$1{\cdot}4 \pm 0{\cdot}3$	$1{\cdot}5 \pm 0{\cdot}2$
D	$-4{\cdot}24 \pm 0{\cdot}015$	$-4{\cdot}37 \pm 0{\cdot}015$

Thus it appears that the singular term does not occur as a simple factor; it is not surprising, therefore, that Padé analysis based on the available series does not yield precise results.

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